43-1-6

APPLIED MATHEMATICS AND STATISTICS LABORATORIES
STANFORD UNIVERSITY
CALIFORNIA

ESTIMATING MISSILE RELIABILITY

BY

S. BLUMENTHAL and J. DENTON

TECHNICAL REPORT NO. 81
October 26, 1962

PREPARED UNDER CONTRACT Nonr-225(52)
(NR-342-022)
FOR
OFFICE OF NAVAL RESEARCH

ESTIMATING MISSILE RELIABILITY

by

S. Blumenthal and J. Denton

TECHNICAL REPORT NO. 81

October 26, 1962

PREPARED UNDER CONTRACT Nonr-225(52)

(NR-342-022)

FOR

OFFICE OF NAVAL RESEARCH

APPLIED MATHEMATICS AND STATISTICS LABORATORIES

STANFORD UNIVERSITY

STANFORD, CALIFORNIA

ESTIMATING MISSILE RELIABILITY

by

S. Blumenthal and J. Denton

I. Introduction.

The problem to be discussed is a specialization of a problem mentioned by Deemer and Mayberry [1961]. Their problem concerned the allocation to targets of a stockpile of missiles, the assignment being made on the basis of the outcome of a testing program designed to estimate missile reliability. Thus the question of how many missiles from the stockpile should be expended in operational test firings must be studied. Somewhat more specifically their problem may be put into the following form. We assume we have a stockpile of missiles and a set of targets $\{T_1, T_2, \ldots, T_t\}$ over which we wish to allocate these missiles in an optimal way. If a missile is completely reliable then it has a certain probability P_j of destroying a target T_j at which it is aimed. Since, however, missiles are not completely reliable, we assume that a missile has reliability R and that the probability that the target T_j survives a single missile is $(1-P_jR)$, while the probability that it survives n missiles is $(1-P_jR)^n$. However, the value of R must be estimated from the operational testing. It is desirable that each target upon which missiles are expended should have enough missiles allocated to it so that its survival probability is very small. However, we wish to avoid assigning more missiles than necessary to a given target. Nevertheless, a given amount of over-assignment is to be preferred to the same amount of under-assignment; i.e., in terms of a non-negative loss function, the loss is zero when the target does not survive and positive otherwise.

II. The Problem.

The specific problem to be discussed below may be formulated as follows. We imagine a circle of fixed radius ρ surrounding each target with the property that if a missile detonates within this circle the target will be destroyed, but the target is undamaged by a missile detonation outside this circle. We associate with each missile a number R [1] called its reliability which we define as the probability that the missile, when aimed at a given target, will detonate within the circle of radius ρ surrounding that target. Thus the probability that a given target survives one missile is

$$1 - R ,$$

and the probability that it survives n missiles is

$$(1 - R)^n ,$$

so the probability of destroying the target with n missiles is

$$1 - (1 - R)^n .$$

Let us confine our attention to the target T. Given a number Q, specified in advance, with $0 < Q < 1$, the number of missiles n_T to be expended on the target T is then

---

[1] The analysis based on R is conceptually the same as if R is replaced by the value $P_j R$ developed in the Introduction.

(1) $$n_T = \min\{h : 1-(1-R)^h \geq Q\} .$$

Since R is unknown, e cannot determine n_T. We therefore wish to obtain an estimate $\hat{R}$ from which n_T may be estimated in the obvious way:

(2) $$\hat{n}_T = \min\{h : 1-(1-\hat{R})^h \geq Q\} .$$

The loss function to be employed in this situation takes the value zero if T is destroyed with probability greater than or equal to Q and one if this probability is less than Q. This leads to the search for an estimator $\hat{R}$ which underestimates R with predetermined probability. That is, given α, $0 < \alpha < 1$, we seek an $\hat{R}$ with the property that

$$P[\hat{R} \leq R] \geq 1-\alpha.$$

In classical terminology $\hat{R}$ is a lower confidence bound for the value of R.

We also assume that the target lies at the origin $(0,0)$ of a Cartesian coordinate system. If $(x_1,y_1),\ldots,(x_n,y_n)$ are the points of impact of the n missiles, then each is thought of as a random observation of a bivariate normal variable with mean vector $(0,0)$ (there is no aiming bias) and covariance matrix Σ given by

$$\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} .$$

In words, X and Y are independently normally distributed with zero means and unknown variances σ_1^2 and σ_2^2. The reliability R then has the form

$$R = R(\Sigma) = P\{X^2 + Y^2 \leq \rho^2\} \tag{3}$$

where ρ is a given positive constant.

We now consider several methods for obtaining confidence intervals for R. Once these are obtained we employ their lower bounds as values $\hat{R}$ to be substituted into equation (2) to obtain the estimate $\hat{n}_T$.

III. Methods of Estimation.

Method 1

On the basis of the sample $(x_1,y_1),\ldots,(x_n,y_n)$ we want a value $\hat{R}$ such that when R is the true value of the parameter, the probability that $\hat{R}$ is no greater than R should be at least $(1-\alpha)$:

$$P\{\hat{R} \leq R|R\} \geq 1-\alpha . \tag{4}$$

The experimenter is to specify α. By choosing $\hat{R}$ in this manner, we are essentially "underestimating" R which leads to a conservative assignment of missiles.

We can always satisfy (4) by setting $\hat{R} = 0$, but this leads to assigning all missiles to one target. Thus we should like to employ an estimator $\hat{R}$ which satisfies (4) while maximizing the number of targets attacked. In classical terminology, this means that we are looking for a shortest upper confidence interval for R.

Let us now confine our attention to the case where $\sigma_1 = \sigma_2 = \sigma$. In this case we have

$$R = \frac{1}{2\pi\sigma^2} \iint_{x^2+y^2 \le \rho^2} e^{-\frac{x^2+y^2}{2\sigma^2}} \, dx\, dy = 1-e^{-\frac{\rho^2}{2\sigma^2}} . \tag{5}$$

It is clear that R is a monotone decreasing function of σ, so that it suffices to find an upper confidence bound $\hat{\sigma}$ for σ, i.e., the desired bound $\hat{R}$ is

$$\hat{R} = 1 - e^{-\rho^2/2\hat{\sigma}^2} . \tag{6}$$

Consider the following hypothesis testing problem:

Given $\mu_1 = \mu_2 = 0$, to test

$$H_0 : \sigma^2 \ge \sigma_0^2$$

$$\text{vs. } H_1 : \sigma^2 < \sigma_0^2 .$$

It is well known that (c.f. Lehmann, 1959) the test which accepts H_0 when $\sum_{i=1}^{n} \frac{(X_i^2+Y_i^2)}{\sigma^2} \ge C_1$ is a uniformly most powerful unbiased test of H_0 vs. H_1 with size α, where C_1 is the upper $100(1-\alpha)\%$ point of the chi-squared distribution with $2n$ degrees of freedom. From this, we see that a "uniformly most accurate" upper confidence bound $\hat{\sigma}^2$, for σ^2 is given by

$$\hat{\sigma}^2 = \frac{1}{C_1} \sum_{i=1}^{n} (X_i^2 + Y_i^2) . \tag{7}$$

Note that the bound given by (7) and (6) has the following property. Among all bounds $\hat{R}$ satisfying (4), this bound minimizes $E_R L(R,\hat{R})$

where $L(R,\hat{R})$ is any measure of the loss resulting from overestimating R, and where $L(R,\hat{R})$ is non-negative for $\hat{R} > R$, and is non-decreasing in $\hat{R}$ (see Lehmann [1959], p. 78). $\hat{n}_T$ is then obtained from (2).

Now let us turn our attention to the case where $\sigma_2^2 = K\sigma_1^2$, where K is a known positive constant. In this case,

$$(8) \qquad R = \frac{1}{2\pi\sigma_1\sigma_2} \iint_{x^2+y^2<\rho^2} e^{-\left(\frac{x^2}{2\sigma_1^2}+\frac{y^2}{2\sigma_2^2}\right)} dx\,dy$$

$$= 1 - \frac{2}{\pi}\int_0^{\pi/2} e^{-\frac{\rho^2}{\sigma_1^2(1+(K-1)\sin^2\theta)}} d\theta .$$

Again, it can be seen that R is a monotone decreasing function of σ_1^2, and again an upper confidence bound $\hat{\sigma}_1^2$ for σ_1^2 yields the lower confidence bound $\hat{R}$, where

$$(9) \qquad \hat{R} = 1 - \frac{2}{\pi}\int_0^{\pi/2} e^{-\frac{\rho^2}{\hat{\sigma}_1^2(1+(K-1)\sin^2\theta)}} d\theta .$$

This last integral may have to be evaluated by numerical methods.

Proceeding exactly as in the case $\sigma_1 = \sigma_2 = \sigma$, we find the following most accurate upper confidence bound $\hat{\sigma}_1^2$ for σ_1^2:

$$(10) \qquad \hat{\sigma}_1^2 = \frac{1}{C_1}\sum_{i=1}^{n}\left[X_i^2 + \left(\frac{Y_i}{K}\right)^2\right],$$

where C_1 is the upper $(1-\alpha)\%$ point of the chi-square distribution with 2n degrees of freedom.

The $\hat{R}$ obtained here has all of the desirable characteristics which were enjoyed by the estimator in the previous case. However, it remains to assign a value to K. This might be done on the basis of previous experience, or by performing preliminary experiments to test an hypothesis of the form $H: K = K_0$ where K_0 is chosen arbitrarily ($K_0 = 1$, say) or on the basis of how large a deviation from unity is significant.

Extension of this method to the case where K is unknown and no assumptions are made concerning it does not appear to be feasible since we may then formulate no hypothesis corresponding to $H: \sigma^2 \geq \sigma_0^2$ for which a UMP unbiased test exists. A similar difficulty intrudes in the case of non-zero covariances.

Method 2

For a sequence of m test firings let the random variable X_i be one or zero according as the i^{th} detonation is or is not inside the circle of radius ρ surrounding the target $(i=1,2,\ldots,m)$. Following Lehmann (1959) set

$$Y = \sum_{i=1}^{m} X_i + U , \tag{11}$$

where U is independent of X_i, $i=1,\ldots,m$, and has a uniform distribution on $(0,1)$. (The use of the statistic Y is equivalent to randomization.) Then Y has probability density

$$\binom{m}{[y]} R^{[y]}(1-R)^{m-[y]} , \quad 0 \leq y < m+1 , \tag{12}$$

where $[y]$ denotes the greatest integer less than or equal to y. The conditions of Lehmann's Corollary 3 (Chapter 8 §5) are then satisfied and $\hat{R}$ is then the solution of

(13) $$\Pr_R[Y > y] = \alpha ,$$

where y is the observed value of Y. A solution exists for $\alpha \leq y \leq m+\alpha$. For $m + \alpha < y$ we take $\hat{R} = 1$, and for $y < \alpha$ we take $\hat{R} = 0$. This bound is then uniformly most accurate in the sense of Lehmann and has the further desirable property of minimizing $E_R L(R,\bar{R})$ subject to the requirement

(14) $$\Pr_R[\bar{R} \leq R] \geq 1-\alpha \quad \text{for all } R.$$

For large samples the usual normal approximation with continuity correction utilizing the statistic

(15) $$Y' = \sum_{i=1}^{m} X_i$$

may be employed. Then $\hat{R}$ is the value of p satisfying

(16) $$\Pr[\frac{Y'}{m} - p) \geq a \sqrt{mp(1-p)}] = 1-\alpha ,$$

where Φ is the cdf of the standard normal distribution and $\Phi(a) = \alpha$.

Method 3

Since it is not possible to find an exact test of the hypothesis: H_0: $R = R_0$ vs. H_1: $R < R_0$ (whose acceptance region provides a lower bound on R); and since it is very difficult to find an approximate test of this hypothesis (at least using the method to be described) we shall describe a large sample test of H_0: $R = R_0$ vs. H_1: $R \neq R_0$. From the acceptance region, we get upper and lower confidence bounds $(\hat{R}_U, \hat{R}_u)$ on R and we can then take $\hat{R}_L$ as a lower confidence bound. If α is the significance level of the 2-sided test, and it is symmetric then

$1-\frac{\alpha}{2}$ should be the confidence level of the interval $(\hat{R}_L, 1)$. Since the test we use is not UMP, the bound obtained will not be sharp. How this bound compares with those obtained by the other methods investigated is not known, nor is the actual confidence level for a small sample, since asymptotic distribution results are used in deriving the test. Now if $\hat{\sigma}_1$ and $\hat{\sigma}_2$ represent upper confidence limits of coefficient $(1-(1-\alpha)^{1/2})$, then $R(\hat{\sigma}_1, \hat{\sigma}_2)$ is an α-level bound on R. This is the crudest and simplest, and it is not clear that the bound $\hat{R}_L$, described below, will be larger than $R(\hat{\sigma}_1, \hat{\sigma}_2)$.

The missile landing coordinates X and Y are independent $N(0, \sigma_1^2)$, $N(0, \sigma_2^2)$ respectively. We want a bound for

$$R = \frac{1}{2\pi\sigma_1\sigma_2} \iint_{x^2+y^2 \leq \rho^2} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_1^2} + \frac{y^2}{\sigma_2^2}\right)} \, dx \, dy \, .$$

Let $U = \sum_{i=1}^{n} x_i^2$, $V = \sum_{i=1}^{n} y_i^2$, and let $\sigma_1^2 = \theta_1$, $\sigma_2^2 = \theta_2$. The Max Likelihood Estimates of θ_1 and θ_2 are $\frac{u}{n}$ and $\frac{v}{n}$ respectively, where u and v are sufficient statistics for θ_1 , θ_2 and have the distribution $p(u,v,\theta_1,\theta_2) = [C_n(uv)^{n/2-1}(\theta_1,\theta_2)^{-n/2}e^{-1/2(\frac{u}{\theta_1} + \frac{v}{\theta_2})}]$.

Let θ_1^0 and θ_2^0 be the maximum likelihood estimates of θ_1, θ_2 under the restriction $R = R_0$. Then we know (see Lehmann (1959), pp. 310 - 311) that under H_o (i.e., when $R = R_0$) that $(-2 \log \Lambda_n)$ (where Λ_n is the "likelihood ratio")

$$\Lambda_n = (uv)^{\frac{n}{2}} n^{-n} e^n (\theta_1^0 \theta_2^0)^{-\frac{n}{2}} \exp(-\tfrac{1}{2}(\frac{u}{\theta_1^0} + \frac{v}{\theta_2^0}))$$

has for large n approximately a χ_1^2 distribution. If $P\{\chi_1^2 \leq C_\eta\} = 1-\eta$, then we would accept H_o if $(-2 \log \Lambda_n) \leq C_\alpha$.

Note that θ_1^0 and θ_2^0 are the solutions of

(i) $\quad \frac{d}{d\theta_1} \quad (p(u,v,\theta_1,\theta_2) + \mu R(\theta_1,\theta_2)) = 0$

(ii) $\quad \frac{d}{d\theta_2} \quad (p(u,v,\theta_1,\theta_2) + \mu R(\theta_1,\theta_2)) = 0$

and $R(\theta_1,\theta_2) = R_o$.

Looking only at (i) and (ii), we see that θ_1^0, θ_2^0 can be found as functions of μ, so $(-2 \log \Lambda_n)$ is a function of μ, say $h(\mu)$. We can show that $h(0) = 0$ and $h(-\infty) = h(+\infty) = \infty$. Thus, if $h(\mu)$ is decreasing for $\mu < 0$ and increasing for $\mu > 0$, there will be exactly two values of μ so that $h(\mu) = C_\alpha$, i.e. $h(\mu_1) = h(\mu_2) = C_\alpha$ $(\mu_1 < 0 < \mu_2)$, and $h(\mu) < C_\alpha$ for $\mu_1 < \mu < \mu_2$. Therefore, we would accept H_o for any μ in this range. Since $\hat{\theta}_1, \hat{\theta}_2$, the solutions of (i) and (ii) are functions of μ, $R(\hat{\theta}_1,\hat{\theta}_2)$ is a function $R(\mu)$ of μ. <u>If</u> R <u>is a monotone (increasing) function of</u> μ, then $\mu_1 < \mu < \mu_2$ corresponds to $R(\mu_1) < R(\mu) < R(\mu_2)$, and H_o would be accepted for $R(\mu_1) < R(\hat{\theta}_1,\hat{\theta}_2) < R(\mu_2)$ where $\hat{\theta}_1, \hat{\theta}_2$ are the solutions of (i) and (ii) under the restriction $R(\theta_1,\theta_2) = R_o$. For large n, this test has size α. It follows that a confidence interval of coefficient $(1-\alpha)$ is given by $(R(\mu_1), R(\mu_2))$, and a lower confidence bound by $R(\mu_1)$.

The above reasoning is based on the assumption that $R(\mu)$ is monotone increasing and that $h(\mu)$ has the properties described. Since $\frac{d}{d\mu} h(\mu) = c\,\mu \frac{d}{d\mu} R(\mu)$, it follows that $h(\mu)$ behaves properly if and only if $R(\mu)$ is monotone increasing. Thus it only remains to show this last monotonicity. It appears to be but this has not yet been demonstrated.

One obvious difficulty about comparing this bound $R(\mu_1)$ with other possible bounds is that it is not possible to solve explicitly for it, and a numerical solution of the equations (i), (ii) and $h(\mu) = C_\alpha$ is necessary.

Method 4

For another large sample procedure for obtaining a lower confidence bound on R when the X and Y aiming errors are uncorrelated and both means are zero, we may adopt the following approach. Let

$$R_o \equiv R = \frac{1}{2\pi\sigma_1\sigma_2} \iint_{x^2+y^2 \le \rho^2} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_1^2} + \frac{y^2}{\sigma_2^2}\right)} dx\,dy$$

$$s_1^2 = \frac{1}{n-1} \sum_{i=1}^{n} x_i^2 , \quad s_2^2 = \frac{1}{n-1} \sum_{i=1}^{n} y_i^2 ,$$

$$\hat{R} = \frac{1}{2\pi s_1 s_2} \iint_{x^2+y^2 \le \rho^2} e^{-\frac{1}{2}\left(\frac{x^2}{s_1^2} + \frac{y^2}{s_2^2}\right)} dx\,dy$$

$$R_1 \equiv \left.\frac{\partial\hat{R}}{\partial s_1^2}\right|_{s_1=\sigma_1} = \frac{1}{8\pi\sigma_1^3\sigma_2} \int\int_{x^2+y^2\leq\rho^2} \left(\frac{x^2\sigma_2-2}{4}\right) e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_1^2}+\frac{y^2}{\sigma_2^2}\right)} dx\, dy$$

$$R_2 \equiv \left.\frac{\partial\hat{R}}{\partial s_2^2}\right|_{s_2=\sigma_2} = \frac{1}{8\pi\sigma_1\sigma_2^3} \int\int_{x^2+y^2\leq\rho^2} \left(\frac{x^2\sigma_2-2}{4}\right) e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_1^2}+\frac{y^2}{\sigma_2^2}\right)} dx\, dy.$$

Employing the constant function 1 as the dominating function, a straightforward application of Lebesgue's dominated convergence theorem shows that R is continuous in s_1^2 and s_2^2. The continuity of $\frac{\partial\hat{R}}{\partial s_1^2}$, $\frac{\partial\hat{R}}{\partial s_2^2}$, $\frac{\partial^2\hat{R}}{\partial s_1^2\partial s_2^2}$, $\frac{\partial^2\hat{R}}{(\partial s_2^2)^2}$ and $\frac{\partial^2\hat{R}}{(\partial s_2^2)^2}$ follows similarly. (The justification for differentiating under the integral sign is found in a theorem on page 67 of Math. Meth. of Stat. by H. Cramér). Thus $\hat{R}$ is asymptotically normal since the conditions of the following theorem of Cramér are satisfied:

Theorem (Cramér p. 366, with suitable notation changes). If in some neighborhood of the point $s_1^2 = \sigma_1^2$, $s_2^2 = \sigma_2^2$ the function $\hat{R}$ is continuous and has continuous derivatives of the first and second order with respect to the arguments s_1^2 and s_2^2, the random variable $\hat{R}$ is asymptotically normal, the mean and variance of the limiting distribution being given by R and $2(\sigma_1^2)^2 R_1^2 + 2(\sigma_2^2)^2 R_2^2 \equiv v$ respectively. Thus $\sqrt{n}\ [\frac{\hat{R}-R}{v}]$ has a limiting $N(0,1)$ distribution. Now set

$$\hat{v} = 2(s_1^2)^2 \frac{\partial\hat{R}}{\partial s_1^2} + 2(s_2^2)^2 \frac{\partial\hat{R}}{\partial s_2^2} .$$

We know that $\frac{\partial \hat{R}}{\partial s_1^2}$ and $\frac{\partial \hat{R}}{\partial s_2^2}$ are continuous in s_1^2 and s_2^2 and that $s_1^2 \to \sigma_1^2$ and $s_2^2 \to \sigma_2^2$ in probability. Therefore $\frac{\partial \hat{R}}{\partial s_1^2} \to R_1$ and $\frac{\partial \hat{R}}{\partial s_2^2} \to R_2$ in probability. So by a theorem of Slutsky (p. 255, Cramér) $\hat{v} \to v$ in probability. We now have recourse to one more pertinent theorem:

Theorem (Cramér §20.6, p. 254). Let $\xi_1, \xi_2, \ldots,$ be a sequence of random variables, with the d.f.s. $F_1, F_2, \ldots$. Suppose that $\{F_n(x)\}$ tends to a d.f. $F(x)$ as $n \to \infty$.

Let $\eta_1, \eta_2, \ldots,$ be another sequence of random variables, and suppose that $\{\eta_n\}$ converges in probability to a constant c. Put

$$X_n = \xi_n + \eta_n \ , \quad Y_n = \xi_n \eta_n \ , \quad Z_n = \frac{\xi_n}{\eta_n} \ .$$

Then the d.f. of X_n tends to $F(x-c)$. Further, if $c > 0$, the d.f. of Y_n tends to $F(\frac{x}{c})$, while the d.f. of Z_n tends to $F(cx)$.

Now

$$\frac{\hat{R}-R}{\hat{v}} = \frac{\hat{R}}{\hat{v}} - \frac{R}{\hat{v}} \ .$$

By Slutsky's theorem $\frac{R}{\hat{v}} \to \frac{R}{v}$ in probability as $n \to \infty$. By Cramér's §20.6 theorem the d.f. of $\frac{\hat{R}}{\hat{v}}$ tends to the d.f. of $\frac{\hat{R}}{v}$. So, by a second application of this theorem the d.f. of $\frac{\hat{R}-R}{\hat{v}}$ tends to the distribution of $\frac{\hat{R}-R}{v}$, i.e. $\sqrt{n} \ (\frac{\hat{R}-R}{\hat{v}})$ has a limiting $N(0,1)$ distribution. Therefore we may proceed to construct a large-sample lower α-confidence bound as follows

$$\Pr[\sqrt{n} \ (\frac{\hat{R}-R}{\hat{v}}) \leq b] = \Phi(b)$$

$$\Pr[\hat{R} - \frac{\hat{v}}{\sqrt{n}} \ b \leq r] = \Phi(b)$$

where Φ is the standard normal c.d.f. and b is chosen so that $\Phi(b) = 1-\alpha$.

We remark that though the derivation might be long and tedious, the extension of this method to the case of non-zero covariance appears to require nothing new in principle.

IV. Discussion.

When $\sigma_1^2 = \sigma_2^2 = \sigma^2$ and the covariances are equal then if $\hat{R}_1$ is the estimator of R given by M1 (method 1) and $\hat{R}_2$ the estimator of R given by M2, we have, since $\hat{R}_2$ is uniformly most accurate

$$R' < R \Rightarrow \Pr\{\hat{R}_2 \leq R'\} \leq \Pr\{\hat{R}_1 \leq R'\} .$$

We note that $\hat{R}_2$ enjoys its pleasant properties independent of the form of the distribution function for the points of missile impact (i.e. whether or not we assume the coordinates of the points of impact follow a bivariate normal distribution). In particular this is true in the case $\sigma_1^2 \neq \sigma_2^2$ when a bivariate normal distribution is assumed. Also, for large samples the estimator based on Y' appears to involve less computational labor than those introduced in M3 and M4.

At present, because of the approximations involved in the estimators, it seems that the only direct way of comparing M3 and M4 with each other as well as with M1 and M2 appears to be by use of a Monte Carlo method.

With respect to the formulation of the problem employed it may be remarked that the form of the loss function requires assigning enough missiles on each target attacked to insure its destruction before including additional targets among those to be attacked. If some targets require more missiles than others, then with a limited missile stockpile those requiring fewest missiles should receive first assignment and so on until all of the missiles in the stockpile are assigned. If all targets require the same number of missiles but are not equally important, this may be reflected by assigning higher priority targets a loss greater than 1, if sufficient missiles are not assigned to insure destruction with probability greater than or equal to Q.

It might be noted that the model set fouth in Section II may be modified in several ways without materially changing the estimation problem involved. As an example we suggest a cumulative damage model in which the target is surrounded by a circle C_T of fixed radius ρ. Each missile causes total destruction inside a circle C_M or radius r about its point of impact. The target is considered to the totally destroyed when all of the area inside the circle C_T is destroyed by missile blasts, i.e. when all of the area of C_T can be covered by the area of circles of radius r drawn about the points of impact of missiles which have been fired at the target. If, once again, we assume that the coordinates of the points of impact are independently distributed and have a bivariate normal distribution about the point of aim with $\sigma_1^2 = \sigma_2^2 = \sigma^2$, then questions about the number of missiles to be expended to obtain a given percent damage with a certain level of confidence may be answered by obtaining an estimate for σ^2. We might alternatively wish to fire enough missiles so that the expected coverage of the circle C_T was greater than some specified amount (see Morganthaler [1961]). In this situation it might also be well to consider whether the point of aim should coincide with the target for every missile in order to obtain maximum coverage from the missiles expended.

REFERENCES

[1] Cramer, H. (1946), Mathematical Methods of Statistics, Princeton University Press, Princeton, New Jersey.

[2] Deemer, Walter, and Mayberry, John, (1961), Application of Statistical Decision Theory to a Missile Testing Problem. Operations Analysis Paper No. 5, Operations Analysis Office, Hqs. U. S. Air Force, Washington, D. C.

[3] Lehmann E. L. (1959), Testing Statistical Hypotheses, John Wiley and Sons, New York.

[4] Morganthaler, G. W. (1961), "Some circular coverage problems", Americal Mathematical Monthly, 48, p. 313.

STANFORD UNIVERSITY
TECHNICAL REPORTS DISTRIBUTION LIST
CONTRACT Nonr-225(52)

Armed Services Technical Information Agency Arlington Hall Station Arlington 12, Virginia	10
Bureau of Supplies and Accounts Code OW Department of the Navy Washington 25, D. C.	1
Head, Logistics and Mathematical Statistics Branch Office of Naval Research Code 436 Washington 25, D. C.	3
Commanding Officer Office of Naval Research Branch Office Navy No. 100, Fleet P. O. New York, N. Y.	2
Commanding Officer Office of Naval Research Branch Office 1000 Geary Street San Francisco 9, California	1
Commanding Officer Office of Naval Research Branch Office 10th Floor, The John Crerar Library Bldg. 86 East Randolph Street Chicago 1, Illinois	1
Commanding Officer Office of Naval Research Branch Office 346 Broadway New York 13, N. Y.	1
Commanding Officer Diamond Ordnance Fuze Labs. Washington 25, D. C.	1
Commanding Officer Picatinny Arsenal (ORDBB-TH8) Dover, New Jersey	1
Commanding Officer Watertown Arsenal (OMRO) Watertown 72, Massachusetts	1
Commanding Officer Attn: W. A. Labs Watertown Arsenal Watertown 72, Massachusetts	1
Commanding Officer Watervliet Arsenal Watervliet, New York	1
Commanding Officer Attn: Inspection Division Springfield Armory Springfield, Massachusetts	1
Commanding Officer Signal Corps Electronic Research Unit, EDL 9560 Technical Service Unit P.O. Box 205 Mountain View, California	1
Commanding Officer 9550 Technical Service Unit Army Liaison Group, Project Michigan Willow Run Research Center Ypsilanti, Michigan	1
Commanding Officer Engineering Research & Development Labs. Fort Belvoir, Virginia	1
Commanding Officer Frankford Arsenal Library Branch, 0270, Bldg. 40 Bridge and Tacony Streets Philadelphia 37, Pennsylvania	1
Commanding Officer Rock Island Arsenal Rock Island, Illinois	1
Commanding General Redstone Arsenal (ORDDW-QC) Huntsville, Alabama	1
Commanding General White Sands Proving Ground (ORDBS-TS-TIB) Las Cruces, New Mexico	1
Commanding General Attn: Paul C. Cox, Ord. Mission White Sands Proving Ground Las Cruces, New Mexico	1
Commanding General Attn: Technical Documents Center Signal Corps Engineering Laboratory Fort Monmouth, New Jersey	1
Commanding General Ordnance Weapons Command Attn: Research Branch Rock Island, Illinois	1
Commanding General U.S. Army Electronic Proving Ground Fort Huachuca, Arizona Attn: Technical Library	1
Commander Wright Air Development Center Attn: ARL Tech. Library, WCRR Wright-Patterson Air Force Base, Ohio	1
Commander Western Development Division, WDSIT P.O. Box 262 Inglewood, California	1
Chief, Research Division Office of Research & Development Office of Chief of Staff U.S. Army Washington 25, D. C.	1
Chief, Computing Laboratory Ballistic Research Laboratory Aberdeen Proving Ground, Maryland	1
Director National Security Agency Attn: REMP-1 Fort George G. Meade, Maryland	2
Director of Operations Operations Analysis Div., AFOOP Hq., U.S. Air Force Washington 25, D. C.	1
Director Snow, Ice & Permafrost Research Establishment Corps of Engineers 1215 Washington Avenue Wilmette, Illinois	1
Director Lincoln Laboratory Lexington, Massachusetts	1
Department of Mathematics Michigan State University East Lansing, Michigan	1
Document Library U.S. Atomic Energy Commission 19th and Constitution Aves., N.W. Washington 25, D. C.	1
Headquarters Oklahoma City Air Materiel Area United States Air Force Tinker Air Force Base, Oklahoma	1
Institute of Statistics North Carolina State College of A & E Raleigh, North Carolina	1
Jet Propulsion Laboratory California Institute of Technology Attn: A.J. Stosick 4800 Oak Grove Drive Pasadena 3, California	1
Librarian The RAND Corporation 1700 Main Street Santa Monica, California	1
Library Division Naval Missile Center Command U.S. Naval Missile Center Attn: J. L. Nickel Point Mugu, California	1
Mathematics Division Code 5077 U.S. Naval Ordnance Test Station China Lake, California	1
NASA Attn: Mr. E.B. Jackson, Office of Aero Intelligence 1724 F Street, N. W. Washington 25, D. C.	1
National Applied Mathematics Labs. National Bureau of Standards Washington 25, D. C.	1
Naval Inspector of Ordnance U.S. Naval Gun Factory Washington 25, D. C. Attn: Mrs. C. D. Hock	1
Office, Asst. Chief of Staff, G-4 Research Branch, R & D Division Department of the Army Washington 25, D. C.	1
Superintendent U.S. Navy Postgraduate School Monterey, California Attn: Library	1
Technical Information Officer Naval Research Laboratory Washington 25, D. C.	6
Technical Information Service Attn: Reference Branch P.O. Box 62 Oak Ridge, Tennessee	1
Technical Library Branch Code 234 U.S. Naval Ordnance Laboratory Attn: Clayborn Graves Corona, California	1
Institute for Defense Analyses Communications Research Division von Neumann Hall Princeton, New Jersey	1

August, 1962

Mr. Irving B. Altman Inspection & QC Division Office, Asst. Secretary of Defense Room 2B870, The Pentagon Washington 25, D. C.	1
Professor T. W. Anderson Department of Statistics Columbia University New York 27, New York	1
Professor Robert Bechhofer Dept. of Industrial and Engineering Administration Sibley School of Mechanical Engineering Cornell University Ithaca, New York	1
Professor Fred. C. Andrews Department of Mathematics University of Oregon Eugene, Oregon	1
Professor Z. W. Birnbaum Department of Mathematics University of Washington Seattle 5, Washington	1
Dr. David Blackwell Department of Mathematical Sciences University of California Berkeley 4, California	1
Professor Ralph A. Bradley Department of Statistics Florida State University Tallahassee, Florida	1
Dr. John W. Cell Department of Mathematics North Carolina State College Raleigh, North Carolina	1
Professor William G. Cochran Department of Statistics Harvard University 2 Divinity Avenue, Room 311 Cambridge 38, Massachusetts	1
Miss Besse B. Day Bureau of Ships, Code 342D Room 3210, Main Navy Department of the Navy Washington 25, D. C.	1
Dr. Walter L. Deemer, Jr. Operations Analysis Div., DCE/D Hq., U.S. Air Force Washington 25, D. C.	1
Professor Cyrus Derman Dept. of Industrial Engineering Columbia University New York 27, New York	1
Dr. Donald P. Gaver Westinghouse Research Labs. Beulah Rd. - Churchill Boro. Pittsburgh 35, Pa.	1
Mr. Harold Gumbel Head, Operations Research Group Code 01-2 Pacific Missile Range Box 1 Point Mugu, California	1
Dr. Ivan Hershner Office, Chief of Research & Dev. U.S. Army, Research Division 3E382 Washington 25, D. C.	1
Professor W. Hirsch Institute of Mathematical Sciences New York University New York 3, New York	1
Mr. Eugene Hixson Code 600.1 GSFC, NASA Greenbelt, Maryland	1
Professor Harold Hotelling Department of Statistics University of North Carolina Chapel Hill, North Carolina	1
Professor Solomon Kullback Department of Statistics George Washington University Washington 7, D. C.	1
Professor W. H. Kruskal Department of Statistics The University of Chicago Chicago, Illinois	1
Professor Eugene Lukacs Department of Mathematics Catholic University Washington 15, D. C.	1
Dr. Craig Magwire 2954 Winchester Way Rancho Cordova, California	1
Professor G. W. McElrath Department of Mechanical Engineering University of Minnesota Minneapolis 14, Minnesota	1
Dr. Knox T. Millsaps Executive Director Air Force Office of Scientific Research Washington 25, D. C.	1
D. E. Newnham Chief, Ind. Engr. Div. Comptroller Hq., San Bernardino Air Materiel Area USAF, Norton Air Force Base, California	1
Professor Edwin G. Olds Department of Mathematics College of Engineering and Sciences Carnegie Institute of Technology Pittsburgh 13, Pennsylvania	1
Dr. William R. Pabst Bureau of Weapons Room 0306, Main Navy Department of the Navy Washington 25, D. C.	1
Mr. Edward Paulson 72-10 41 Ave. Woodside 77 New York, New York	1
H. Walter Price, Chief Reliability Branch, 750 Diamond Ordnance Fuze Laboratory Room 105, Building 83 Washington 25, D. C.	1
Professor Ronald Pyke Mathematics Department University of Washington Seattle 5, Washington	1
Dr. Paul Rider Wright Air Development Center, WCRRM Wright-Patterson A.F.B., Ohio	1
Professor Herbert Robbins Dept. of Mathematical Statistics Columbia University New York 27, New York	1
Professor Murray Rosenblatt Department of Mathematics Brown University Providence 12, Rhode Island	1
Professor Herman Rubin Department of Statistics Michigan State University East Lansing, Michigan	1
Professor J. S. Rustagi College of Medicine University of Cincinnati Cincinnati, Ohio	1
Professor I. R. Savage School of Business Administration University of Minnesota Minneapolis, Minnesota	1
Miss Marion M. Sandomire 2281 Cedar Street Berkeley 9, California	1
Professor L. J. Savage Mathematics Department University of Michigan Ann Arbor, Michigan	1
Professor W. L. Smith Statistics Department University of North Carolina Chapel Hill, North Carolina	1
Dr. Milton Sobel Statistics Department University of Minnesota Minneapolis, Minnesota	1
Mr. G. P. Steck Division 5511 Sandia Corp., Sandia Base Albuquerque, New Mexico	1
Professor Donald Truax Department of Mathematics University of Oregon Eugene, Oregon	1
Professor John W. Tukey Department of Mathematics Princeton University Princeton, New Jersey	1
Professor G. S. Watson Department of Mathematics University of Toronto Toronto 5, Ontario, Canada	1
Dr. Harry Weingarten Special Projects Office, SP2016 Navy Department Washington 25, D. C.	1
Dr. F. J. Weyl, Director Mathematical Sciences Division Office of Naval Research Washington 25, D. C.	1
Dr. John Wilkes Office of Naval Research, Code 200 Washington 25, D. C.	1
Professor S. S. Wilks Department of Mathematics Princeton University Princeton, New Jersey	1
Mr. Silas Williams Standards Branch, Proc. Div. Office, DC/S for Logistics Department of the Army Washington 25, D. C.	1
Professor Jacob Wolfowitz Department of Mathematics Cornell University Ithaca, New York	1
Mr. William W. Wolman Code MER - Bldg. T-2 Room C301 700 Jackson Place, N. W. Washington 25, D. C.	1
Marvin Zelen Mathematics Research Center U. S. Army University of Wisconsin Madison 6, Wisconsin	1
Additional copies for project leader and assistants and reserve for future requirements	50

Contract Nonr-225(52)
August, 1962

(175)

JOINT SERVICES ADVISORY GROUP

Mr. Fred Frishman
Army Research Office
Arlington Hall Station
Arlington, Virginia 1

Mrs. Dorothy M. Gilford
Logistics and Mathematical Statistics Branch
Office of Naval Research
Washington 25, D. C. 3

Dr. Robert Lundegard
Logistics and Mathematical Statistics Branch
Office of Naval Research
Washington 25, D. C. 1

Mr. R. H. Noyes
Inst. for Exploratory Research
USASRDL
Fort Monmouth, New Jersey 1

Lt. Col. John W. Querry, Chief
Applied Mathematics Division
Air Force Office of Scientific Research
Washington 25, D. C. 1

Major Oliver A. Shaw, Jr.
Mathematics Division
Air Force Office of Scientific Research
Washington 25, D. C. 2

Mr. Carl L. Schaniel
Code 122
U.S. Naval Ordnance Test Station
China Lake, California 1

Mr. J. Weinstein
Institute for Exploratory Research
USASRDL
Fort Monmouth, New Jersey 1